

THE STRUCTURE OF SOBOLEV EXTENSION OPERATORS

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ABSTRACT. Let $L^{m,p}(\mathbb{R}^n)$ denote the Sobolev space of functions whose m -th derivatives lie in $L^p(\mathbb{R}^n)$, and assume that $p > n$. For $E \subseteq \mathbb{R}^n$, denote by $L^{m,p}(E)$ the space of restrictions to E of functions $F \in L^{m,p}(\mathbb{R}^n)$. It is known that there exist bounded linear maps $T : L^{m,p}(E) \rightarrow L^{m,p}(\mathbb{R}^n)$ such that $Tf = f$ on E for any $f \in L^{m,p}(E)$. We show that T cannot have a simple form called “bounded depth.”

1. INTRODUCTION

Let \mathbb{X} denote any of the following standard function spaces on \mathbb{R}^n :

- $\mathbb{X} = C^m(\mathbb{R}^n)$, the space of real-valued $F \in C^m_{\text{loc}}(\mathbb{R}^n)$ for which the norm

$$\|F\|_{C^m(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} \max_{|\alpha| \leq m} |\partial^\alpha F(x)| \text{ is finite;}$$

- $\mathbb{X} = C^{m,s}(\mathbb{R}^n)$, the space of all functions $F \in C^m(\mathbb{R}^n)$ for which the norm

$$\|F\|_{C^{m,s}(\mathbb{R}^n)} := \|F\|_{C^m(\mathbb{R}^n)} + \sup_{\substack{x,y \in \mathbb{R}^n \\ x \neq y}} \max_{|\alpha|=m} \frac{|\partial^\alpha F(x) - \partial^\alpha F(y)|}{|x - y|^s}$$

is finite (here $0 < s < 1$);

- $\mathbb{X} = L^{m,p}(\mathbb{R}^n)$, the homogeneous Sobolev space of all real-valued functions F for which the seminorm

$$\|F\|_{L^{m,p}(\mathbb{R}^n)} := \|\nabla^m F\|_{L^p(\mathbb{R}^n)} \text{ is finite.}$$

(Here, we take $p > n$, so that $\mathbb{X} \subseteq C^{m-1,1-n/p}_{\text{loc}}(\mathbb{R}^n)$, by the Sobolev theorem.)

For $E \subseteq \mathbb{R}^n$, we set $\mathbb{X}(E) := \{F|_E : F \in \mathbb{X}\}$, equipped with the seminorm

$$\|f\|_{\mathbb{X}(E)} := \inf\{\|F\|_{\mathbb{X}} : F \in \mathbb{X}, F = f \text{ on } E\}.$$

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Let $A \geq 1$ be a real number. An extension operator for $\mathbb{X}(E)$ with norm A is a linear map $T : \mathbb{X}(E) \rightarrow \mathbb{X}$ such that for all $f \in \mathbb{X}(E)$ we have

$$Tf = f \text{ on } E$$

and

$$\|Tf\|_{\mathbb{X}} \leq A\|f\|_{\mathbb{X}(E)}.$$

For $\mathbb{X} = C^m(\mathbb{R}^n)$ or $C^{m,s}(\mathbb{R}^n)$ and $E \subseteq \mathbb{R}^n$ arbitrary, there exists an extension operator whose norm depends only on m, n . Similarly, for $\mathbb{X} = L^{m,p}(\mathbb{R}^n)$ and E arbitrary, there exists an extension operator whose norm depends only on m, n, p . See [1, 2, 4].

We want to know whether such extension operators can be taken to have a simple form when E is finite. Recall that any linear map $T : \mathbb{X}(E) \rightarrow \mathbb{X}$ ($E \subseteq \mathbb{R}^n$ finite) has the form

$$Tf(x) = \sum_{y \in E} \lambda(x, y)f(y) \quad (\text{all } x \in \mathbb{R}^n),$$

with coefficients $\lambda(x, y)$ independent of f . Let D be a positive integer. We say that T has depth D if, for each fixed x , at most D of the coefficients $\lambda(x, y)$ are nonzero.

Let $\mathbb{X} = C^m(\mathbb{R}^n)$ or $C^{m,s}(\mathbb{R}^n)$, and let $E \subseteq \mathbb{R}^n$ be finite. Then there exists an extension operator for $\mathbb{X}(E)$, whose norm and depth depend only on m, n . See [1, 3].

Thus, it is natural to ask the following:

Let $\mathbb{X} = L^{m,p}(\mathbb{R}^n)$, and let $E \subseteq \mathbb{R}^n$ be finite. Does there exist an extension operator for $\mathbb{X}(E)$, whose norm and depth depend only on m, n, p ?

Unfortunately, the answer is NO. In this paper, we establish the following result.

Theorem 1. *Let $p > 2$, $A \geq 1$ and $D \geq 1$ be given.*

Then there exists a finite set $E \subseteq \mathbb{R}^2$ such that $L^{2,p}(E)$ has no extension operator of norm A and depth D .

More precisely, for $N \geq 2$, let

$$E_N := \{(2^{-k}, (2^{-k})^{2-2/p}) : k = 2, \dots, N\} \cup \{(0, 0)\} \subseteq \mathbb{R}^2. \quad (1)$$

Theorem 2. *Let $p > 2$, $A \geq 1$, $D \geq 1$, and let $0 < \epsilon < \frac{3}{p}$.*

If $L^{2,p}(E_N)$ has an extension operator with norm A and depth D , then

$$A \cdot D^{5/p} > c(\epsilon, p) \cdot N^\epsilon, \text{ where } c(\epsilon, p) \text{ depends only on } \epsilon \text{ and } p.$$

Theorem 2 will be proven in the next section. Theorem 1 follows at once from Theorem 2.

We mention a few related results in the literature. For $\mathbb{X} = C^{m,s}(\mathbb{R}^n)$, Luli [6] constructed extension operators of bounded depth without the assumption that E is finite. The analogous result for $\mathbb{X} = C^m(\mathbb{R}^n)$ is false; however, there exist extension operators of “bounded breadth.” (See [3].) For $\mathbb{X} = L^{m,p}(\mathbb{R}^n)$ and E finite, an extension operator may be taken to have “assisted bounded depth”; see [4].

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2. PROOF OF THEOREM 2

Fix $p > 2$ and $0 < \epsilon < \frac{1}{3p}$, and let $\alpha := 1 - \frac{2}{p}$. Unless stated otherwise, C, c , etc. denote constants depending only on p , which may change value from one occurrence to the next.

For any C^1 function $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $y \in \mathbb{R}^2$, let $J_y F$ denote the first order Taylor polynomial of F at y :

$$(J_y F)(x) = F(y) + \nabla F(y) \cdot (x - y).$$

We require $p > 2$ so that the Sobolev theorem holds. In particular, after modification on some measure zero subset, each $F \in L^{2,p}(\mathbb{R}^2)$ belongs to $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^2)$ and satisfies the inequalities:

$$\begin{aligned} |\nabla F(x) - \nabla F(y)| &\leq C \|F\|_{L^{2,p}(\mathbb{R}^2)} |x - y|^\alpha \\ |F(x) - J_y F(x)| &\leq C \|F\|_{L^{2,p}(\mathbb{R}^2)} |x - y|^{1+\alpha} \end{aligned} \quad (\text{all } x, y \in \mathbb{R}^2). \quad (2)$$

We extend the $L^{2,p}$ norm to \mathbb{R}^2 -valued functions by setting

$$\|\Psi\|_{L^{2,p}(\mathbb{R}^2)} := \|\Psi_1\|_{L^{2,p}(\mathbb{R}^2)} + \|\Psi_2\|_{L^{2,p}(\mathbb{R}^2)}, \quad \text{where } \Psi = (\Psi_1, \Psi_2) \text{ in coordinates.}$$

We define the curve $\gamma := \{(s, s^{1+\alpha}) : s \in [0, 1]\} \subseteq \mathbb{R}^2$. Let $N \geq 2$. We write E for the subset E_N defined in the introduction:

$$E := \{(2^{-k}, (2^{-k})^{1+\alpha}) : k = 2, \dots, N\} \cup \{(0, 0)\} \subseteq \gamma. \quad (3)$$

In proving Theorem 2, it suffices to assume that N is sufficiently large. More precisely, we henceforth assume that

$$N \geq Z, \text{ where } Z \geq 1 \text{ is some large constant that depends only on } p \text{ and } \epsilon. \quad (4)$$

We determine Z through Lemma 1 below.

Lemma 1. *There exists $Z \geq 1$ depending only on p and ϵ , such that the following holds. Assume (4). Then for any $G \in L^{2,p}(\mathbb{R}^2)$ with*

$$G = 0 \text{ on } E \text{ and } \|G\|_{L^{2,p}(\mathbb{R}^2)} \leq 1,$$

we have $|\nabla G(0)| \leq N^{-\epsilon}$.

Lemma 2. *For any integer $D \geq 1$ and subset $S \subseteq \gamma$ with $\#S \leq D$, there exists $H \in L^{2,p}(\mathbb{R}^2)$ that satisfies*

$$H = 0 \text{ on } S, |\nabla H(0)| \geq 1, \text{ and } \|H\|_{L^{2,p}(\mathbb{R}^2)} \leq C_2 D^{\frac{5}{p}}, \quad (5)$$

where $C_2 = C_2(p)$ depends only on p .

We now prove Theorem 2, presuming the validity of Lemmas 1 and 2. These lemmas are proven later in the section.

In proving Theorem 2, it suffices to assume that (4) holds with Z determined by Lemma 1.

Let $A \geq 1$, $D \geq 1$, and let $T : L^{2,p}(E) \rightarrow L^{2,p}(\mathbb{R}^2)$ be an extension operator with norm A and depth D . In other terms, for any $f : E \rightarrow \mathbb{R}$,

$$Tf = f \text{ on } E, \quad (6)$$

$$\|Tf\|_{L^{2,p}(\mathbb{R}^2)} \leq A\|f\|_{L^{2,p}(E)}, \text{ and} \quad (7)$$

$$Tf(x) = \sum_{y \in E} \lambda(x, y) f(y) \quad \text{for all } x \in \mathbb{R}^2, \quad (8)$$

where the coefficients $\lambda(x, y)$ satisfy

$$\#\{y \in E : \lambda(x, y) \neq 0\} \leq D \quad \text{for all } x \in \mathbb{R}^2. \quad (9)$$

Note that $\lambda(x, y) = (T\delta_y)(x)$, where $\delta_y : E \rightarrow \mathbb{R}$ equals 1 at y , and equals 0 on $E \setminus \{y\}$. Thus, $\lambda(\cdot, y) \in L^{2,p}(\mathbb{R}^2)$ for each fixed $y \in E$. It follows from the Sobolev theorem that the function $x \mapsto \lambda(x, y)$ belongs to $C^1(\mathbb{R}^2)$ for each fixed $y \in E$.

Let

$$S := \{y \in E : \nabla_x \lambda(0, y) \neq 0\}. \quad (10)$$

We claim that $\#S \leq D$. Indeed, suppose for the sake of contradiction that there exist distinct $y_1, \dots, y_{D+1} \in E$ such that $\nabla_x \lambda(0, y_k) \neq 0$ for each $k = 1, \dots, D+1$. Then, by the implicit function theorem, there exists $x \in \mathbb{R}^2$ such that $\lambda(x, y_k) \neq 0$ for each $k = 1, \dots, D+1$. This contradicts (9), hence proving $\#S \leq D$.

Note that $S \subseteq \gamma$ (see (3),(10)). By Lemma 2 there exists $H \in L^{2,p}(\mathbb{R}^2)$ with

$$H = 0 \text{ on } S, \quad |\nabla H(0)| \geq 1, \text{ and } \|H\|_{L^{2,p}(\mathbb{R}^2)} \leq C_2 D^{\frac{5}{p}}. \quad (11)$$

Define $F = T(H|_E)$. From (8),

$$\nabla F(0) = \sum_{y \in E} \nabla_x \lambda(0, y) H(y),$$

For $y \in S$ the summand vanishes because $H = 0$ on S , while for $y \in E \setminus S$ the summand vanishes by definition of S (see (10)). Therefore, $\nabla F(0) = 0$. Finally, (6) implies that $F = H$ on E , while (7) and (11) imply that

$$\|F\|_{L^{2,p}(\mathbb{R}^2)} \leq A \|H|_E\|_{L^{2,p}(E)} \leq A \|H\|_{L^{2,p}(\mathbb{R}^2)} \leq C_2 A D^{\frac{5}{p}}.$$

We define $F_0 := F - H$. From (11) and the above properties of F ,

$$F_0 = 0 \text{ on } E, \quad |\nabla F_0(0)| = |\nabla H(0)| \geq 1, \text{ and } \|F_0\|_{L^{2,p}(\mathbb{R}^2)} \leq (C_2 + 1) A D^{\frac{5}{p}}.$$

Taking $G = F_0 \cdot \left[(C_2 + 1) A D^{\frac{5}{p}} \right]^{-1}$ in Lemma 1, we obtain

$$N^{-\epsilon} \geq |\nabla G(0)| \geq \left[(C_2 + 1) A D^{\frac{5}{p}} \right]^{-1}.$$

This completes the proof of Theorem 2. In the following subsections we prove Lemmas 1 and 2.

2.1. Besov spaces. The Besov seminorm of a differentiable function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is

$$\|\varphi\|_{\dot{B}_p(\mathbb{R})} := \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\varphi'(s) - \varphi'(t)|^p}{|s - t|^p} ds dt \right)^{1/p}.$$

The Besov space $\dot{B}_p(\mathbb{R})$ consists of functions with finite Besov seminorm.

The Besov and Sobolev spaces are related through the following trace/extension theorem (see [7, 8]).

Theorem 3. Let \mathcal{R} denote the restriction operator $\mathcal{R}(F) = F|_{\mathbb{R} \times \{0\}}$, defined for continuous functions $F : \mathbb{R}^2 \rightarrow \mathbb{R}$.

- The restriction operator $\mathcal{R} : L^{2,p}(\mathbb{R}^2) \rightarrow \dot{B}_p(\mathbb{R})$ is bounded. In other terms, $\|\mathcal{R}(G)\|_{\dot{B}_p(\mathbb{R})} \leq C_{SB} \|G\|_{L^{2,p}(\mathbb{R}^2)}$ for every $G \in L^{2,p}(\mathbb{R}^2)$.
- There exists a bounded extension operator $\mathcal{E} : \dot{B}_p(\mathbb{R}) \rightarrow L^{2,p}(\mathbb{R}^2)$. In other terms, $\mathcal{E}(g)|_{\mathbb{R} \times \{0\}} = g$ and $\|\mathcal{E}(g)\|_{L^{2,p}(\mathbb{R}^2)} \leq C_{SB} \|g\|_{\dot{B}_p(\mathbb{R})}$ for any $g \in \dot{B}_p(\mathbb{R})$.

Given $\bar{E} = \{s_1, \dots, s_K\} \subseteq \mathbb{R}$ and $\phi : \bar{E} \rightarrow \mathbb{R}$, where $s_1 < \dots < s_K$, we denote the Besov trace seminorm of ϕ by

$$\|\phi\|_{\dot{B}_p(\bar{E})} := \inf \{ \|\varphi\|_{\dot{B}_p(\mathbb{R})} : \varphi \in \dot{B}_p(\mathbb{R}), \varphi = \phi \text{ on } \bar{E} \}.$$

Let $s_0 := -\infty$ and $s_{K+1} := +\infty$. Define

$$A_{kl} := \int_{s_{k-1}}^{s_k} \int_{s_l}^{s_{l+1}} \frac{1}{|s-t|^p} ds dt \quad (\text{all } 1 \leq k < l \leq K). \quad (12)$$

For $1 \leq k \leq K$, let $n(k) \in \{1, \dots, K\}$ be such that $s_{n(k)} \in \bar{E}$ is a nearest neighbor of s_k , and let

$$m_k := \frac{\phi(s_k) - \phi(s_{n(k)})}{s_k - s_{n(k)}}.$$

For $1 \leq k \leq K-1$, let $\Delta_k := |s_k - s_{k+1}|$, and let

$$M_k := \frac{|m_k - m_{k+1}|}{\Delta_k} + \frac{|\phi(s_k) + m_k \cdot (s_{k+1} - s_k) - \phi(s_{k+1})|}{\Delta_k^2}.$$

The following expression for the Besov trace seminorm can be found in [5] (see Claims 1 and 3 in the proof of Proposition 3.2).

$$c \cdot \|\phi\|_{\dot{B}_p(\bar{E})}^p \leq \sum_{k=1}^{K-1} M_k^p \Delta_k^2 + \sum_{k=1}^{K-1} \sum_{l=k+1}^K |m_k - m_l|^p A_{kl} \leq C \cdot \|\phi\|_{\dot{B}_p(\bar{E})}^p. \quad (13)$$

2.2. Proof of Lemma 1. Recall that $0 < \epsilon < \frac{1}{3p}$. Let $Z \geq 1$ be a parameter, determined before the end of the proof. We assume that (4) holds, that is, $N \geq Z$. In this subsection, constants written C, c , etc. may depend on p, ϵ , but are independent of other parameters.

For the sake of contradiction, suppose that $G \in L^{2,p}(\mathbb{R}^2)$ satisfies

$$\begin{aligned} G &= 0 \text{ on } E = \{(2^{-k}, (2^{-k})^{1+\alpha}) : k = 2, \dots, N\} \cup \{(0, 0)\}, \\ \|G\|_{L^{2,p}(\mathbb{R}^2)} &\leq 1 \quad \text{and} \quad |\nabla G(0)| \geq N^{-\epsilon}. \end{aligned} \quad (14)$$

Furthermore, by renormalizing G we may assume

$$N^{-\epsilon} \leq |\nabla G(0)| \leq 1. \quad (15)$$

Let $\delta := N^{-1/\alpha}$, and let $\theta \in C_0^\infty(\mathbb{R}^2)$ satisfy

$$\begin{aligned} & \text{(a) } \text{supp}(\theta) \subseteq B(0, \delta), \quad \text{(b) } \theta = 1 \text{ on } B(0, \delta/2), \text{ and} \\ & \text{(c) } |\partial^\beta \theta| \leq C\delta^{-|\beta|}, \text{ whenever } |\beta| \leq 2. \end{aligned} \quad (16)$$

Define $H = \theta G + (1 - \theta)J_0 G$. First we use the Liebniz rule, (16.c) and the fact that H is affine on $\mathbb{R}^2 \setminus B(0, \delta)$ (this follows from (16.a)), and then we use the Sobolev theorem (see (2)) and $\|G\|_{L^{2,p}(\mathbb{R}^2)} \leq 1$, obtaining that

$$\begin{aligned} \|H\|_{L^{2,p}(\mathbb{R}^2)} &\leq C \cdot \left(\|G\|_{L^{2,p}(\mathbb{R}^2)} + \delta^{-1} \|\nabla G - \nabla J_0 G\|_{L^p(B(0,\delta))} \right. \\ &\quad \left. + \delta^{-2} \|G - J_0 G\|_{L^p(B(0,\delta))} \right) \leq C'. \end{aligned} \quad (17)$$

From (16.b) and $G = 0$ on E ,

$$H = 0 \text{ on } E \cap B(0, \delta/2). \quad (18)$$

Note that $\nabla H(0) = \nabla G(0)$, thanks to (16.b). Thus, for each $y \in B(0, \delta)$, applying the Sobolev theorem and (17) we obtain

$$|\nabla H(y) - \nabla G(0)| = |\nabla H(y) - \nabla H(0)| \leq C' \|H\|_{L^{2,p}(\mathbb{R}^2)} |y|^\alpha \leq C'' \delta^\alpha = C'' N^{-1}. \quad (19)$$

Note that (19) also holds for $y \in \mathbb{R}^2$, since H is affine on $\mathbb{R}^2 \setminus B(0, \delta)$. Since N is sufficiently large (see (4)) and $\epsilon < 1$, it follows from (15) and (19) that

$$cN^{-\epsilon} \leq |\nabla H(y)| \leq C \text{ for all } y \in \mathbb{R}^2. \quad (20)$$

Note that $H(y_0) = H(y_1) = 0$, where $y_0 := (0, 0)$ and $y_1 := (2^{-N}, 2^{-N(1+\alpha)})$, for N sufficiently large. This follows from (18), since $y_1 \in B(0, N^{-1/\alpha}/2)$ when N is sufficiently large. Thus, for $v := (y_0 - y_1)/|y_0 - y_1|$, the mean value theorem implies that $v \cdot \nabla H(x^*) = 0$ for some $x^* \in B(0, \delta)$ on the line segment joining y_0 and y_1 . By the Sobolev theorem and (17) it follows that

$$|v \cdot \nabla H| \leq C\delta^\alpha = CN^{-1} \text{ on } B(0, \delta).$$

Hence, $|\partial_1 H| \leq C'N^{-1}$ on $B(0, \delta)$, thanks to the upper bound from (20) and the fact $|v - (1, 0)| \leq C2^{-N\alpha}$. Since H is affine on $\mathbb{R}^2 \setminus B(0, \delta)$, we conclude that

$$|\partial_1 H(y)| \leq C'N^{-1} \quad \text{for all } y \in \mathbb{R}^2. \quad (21)$$

Thus, for N sufficiently large, the lower bound in (20) and $\epsilon < 1$ imply that

$$|\partial_2 H(y)| \geq c'N^{-\epsilon} \quad \text{for all } y \in \mathbb{R}^2. \quad (22)$$

We define $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $\Phi(s, t) = (s, H(s, t))$. The diffeomorphism Φ maps onto \mathbb{R}^2 because $|\partial_2 H|$ is bounded away from zero (see (22)). By (20)-(22), $\nabla \Phi(x)$ takes the form

$$\nabla \Phi(x) = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}, \quad \text{where } |a| \leq CN^{-1} \text{ and } cN^{-\epsilon} \leq |b| \leq C. \quad (23)$$

Thus, $\nabla \Phi(x)$ is invertible for each $x \in \mathbb{R}^2$ and

$$[\nabla \Phi(x)]^{-1} = \begin{pmatrix} 1 & 0 \\ \bar{a} & \bar{b} \end{pmatrix}, \quad \text{where } |\bar{a}| \leq \bar{C}N^{\epsilon-1} \text{ and } |\bar{b}| \leq \bar{C}N^{\epsilon}. \quad (24)$$

We now define $\Psi = \Phi^{-1}$, and write $\Phi = (\Phi_1, \Phi_2)$, $\Psi = (\Psi_1, \Psi_2)$ in coordinates. Differentiating twice the identity $\Psi \circ \Phi = \text{Id}$ shows that

$$\nabla \Phi(x) \cdot \nabla^2 \Psi_j(\Phi(x)) \cdot \nabla \Phi(x) = - \sum_{l=1}^2 \nabla^2 \Phi_l(x) \cdot \partial_l \Psi_j(\Phi(x)) \quad (\text{all } x \in \mathbb{R}^2, j \in \{1, 2\}).$$

Now, perform the following operations on the above equation: Multiply through twice by $[\nabla \Phi(x)]^{-1}$ (on the left and right), use the identity $\nabla \Psi(\Phi(x)) = [\nabla \Phi(x)]^{-1}$, substitute $x = \Phi^{-1}(y)$ on both sides, take p^{th} powers, sum over $j \in \{1, 2\}$, integrate over $y \in \mathbb{R}^2$, and perform the change of variable $y = \Phi(x)$ on the right-hand side. Thus, we obtain

$$\|\Psi\|_{L^{2,p}(\mathbb{R}^2)}^p \leq C \|\Phi\|_{L^{2,p}(\mathbb{R}^2)}^p \|\det(\nabla \Phi)\|_{L^\infty} \|(\nabla \Phi)^{-1}\|_{L^\infty}^{3p}. \quad (25)$$

Next, insert into (25) the bounds $\|\det(\nabla \Phi)\|_{L^\infty} \leq C$, $\|(\nabla \Phi)^{-1}\|_{L^\infty} \leq CN^\epsilon$ and $\|\Phi\|_{L^{2,p}(\mathbb{R}^2)} = \|H\|_{L^{2,p}(\mathbb{R}^2)} \leq C'$ obtained from (23),(24) and (17). Thus,

$$\|\Psi\|_{L^{2,p}(\mathbb{R}^2)} \leq CN^{3\epsilon}. \quad (26)$$

Define $\varphi = \Psi_2|_{\mathbb{R} \times \{0\}}$. By (26) and Theorem 3,

$$\|\varphi\|_{\dot{B}_p(\mathbb{R})} \leq C_{SB} \|\Psi_2\|_{L^{2,p}(\mathbb{R}^2)} \leq C'N^{3\epsilon}. \quad (27)$$

It follows from (18) and the definition $\Phi(s, t) = (s, H(s, t))$ that

$$\Phi(E \cap B(0, \delta/2)) \subseteq \mathbb{R} \times \{0\}.$$

In coordinates, $\Psi = \Phi^{-1}$ takes the form $\Psi(u, v) = (u, \Psi_2(u, v))$. Applying Ψ to the previous set containment and using the definition of φ , we obtain

$$E \cap B(0, \delta/2) \subseteq \{(u, \varphi(u)) : u \in \mathbb{R}\}. \quad (28)$$

For some integer $K \geq 0$, we write

$$E \cap B(0, \delta/2) = \{(0, 0), (2^{-N}, 2^{-N(1+\alpha)}), \dots, (2^{K-N}, 2^{(K-N)(1+\alpha)})\}.$$

Thus, $2^{K-N} \geq c\delta$ for some $c > 0$. Since $\delta = N^{-1/\alpha}$, we obtain

$$K \geq N - C \log(N). \quad (29)$$

Let $s_k := 2^{k-N}$ for $k = 1, \dots, K$, and let $\bar{E} := \{s_1, \dots, s_K\}$. Define $\phi : \bar{E} \rightarrow \mathbb{R}$ by $\phi(2^{k-N}) = (2^{k-N})^{1+\alpha}$ for $k = 1, \dots, K$.

Next, we apply (13) for the \bar{E} and ϕ chosen above. The quantity A_{kl} defined in (12) satisfies

$$A_{kl} \geq \int_{2^{k-1-N}}^{2^{k-N}} \int_{2^{l-1-N}}^{2^{l-N}} \frac{1}{|s-t|^p} ds dt \geq c \cdot 2^{-(l-N)p} 2^{k-N} 2^{l-N} \quad (\text{all } 1 \leq k < l \leq K). \quad (30)$$

Thanks to (28), the function φ equals ϕ on \bar{E} . Thus, from (13) and (30),

$$\begin{aligned} \|\varphi\|_{B_p(\mathbb{R})}^p &\geq \|\phi\|_{B_p(\bar{E})}^p \geq c \sum_{k=2}^{K-1} \sum_{l=k+1}^K |m_k - m_l|^p \cdot 2^{-(l-N)p} 2^{k-N} 2^{l-N}, \text{ where} \\ m_i &:= \left[(2^{i-N})^{1+\alpha} - (2^{i-1-N})^{1+\alpha} \right] / [2^{i-N} - 2^{i-1-N}] = (2 - 2^{-\alpha}) \cdot 2^{(i-N)\alpha}. \end{aligned}$$

Note that $|m_k - m_l| \geq c \cdot 2^{(l-N)\alpha}$ for $2 \leq k < l \leq K$. Inserting this inequality in the above equation, and using $\alpha p = p - 2$, we obtain

$$\|\varphi\|_{B_p(\mathbb{R})}^p \geq c' \sum_{k=2}^{K-1} \sum_{l=k+1}^K 2^{(l-N)(p-2)} 2^{-(l-N)p} 2^{k-N} 2^{l-N} \geq c'' \sum_{k=2}^{K-1} 1 = c'' \cdot (K-2).$$

Finally, from (27) and (29), we obtain

$$c''N - C'' \log(N) \leq (C')^p N^{3\epsilon p}.$$

Since $\epsilon < \frac{1}{3p}$, the above inequality gives a contradiction when N is sufficiently large. Thus, (14) cannot hold, completing the proof by contradiction. We now take $Z = Z(\epsilon, p)$ sufficiently large, so that the previous arguments hold for $N \geq Z$. This completes the proof of Lemma 1. \blacksquare

2.3. Proof of Lemma 2. Let $S \subseteq \gamma$ with $\#S \leq D$ be given. For ease of notation, we may assume that $\#S = D$. We must construct an $H \in L^{2,p}(\mathbb{R}^2)$ that satisfies (5). To start, write

$$S = \{(s_1, s_1^{1+\alpha}), \dots, (s_D, s_D^{1+\alpha})\} \text{ with } 0 \leq s_1 < s_2 < \dots < s_D \leq 1.$$

Let $\bar{S} := \{s_1, \dots, s_D\}$, and define $\phi : \bar{S} \rightarrow \mathbb{R}$ by $\phi(s_k) = (s_k)^{1+\alpha}$ for $k = 1, \dots, D$. Next, we apply (13) to this subset \bar{S} and function ϕ .

We first obtain an estimate on A_{kl} defined in (12):

$$A_{kl} \leq \int_{-\infty}^{s_k} \int_{s_l}^{\infty} \frac{1}{|s-t|^p} ds dt \leq C \cdot |s_k - s_l|^{2-p} \quad (\text{all } 1 \leq k < l \leq D). \quad (31)$$

Let $s_{n(k)} \in \bar{S}$ be a nearest neighbor to s_k , for each $1 \leq k \leq D$, and let

$$m_k := \frac{(s_k)^{1+\alpha} - (s_{n(k)})^{1+\alpha}}{s_k - s_{n(k)}}.$$

From (13), (31) and $\alpha p = p - 2$, there exists $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$S \subseteq \{(s, \varphi(s)) : s \in \mathbb{R}\}, \text{ and} \quad (32)$$

$$\begin{aligned} \|\varphi\|_{B_p(\mathbb{R})}^p &\leq C \sum_{k=1}^{D-1} \frac{|(s_k)^{1+\alpha} + m_k \cdot (s_{k+1} - s_k) - (s_{k+1})^{1+\alpha}|^p}{|s_{k+1} - s_k|^{(1+\alpha)p}} \\ &\quad + C \sum_{k=1}^{D-1} \sum_{l=k+1}^D \frac{|m_k - m_l|^p}{|s_k - s_l|^{\alpha p}}. \end{aligned} \quad (33)$$

By the mean value theorem, each m_k takes the form $(1 + \alpha)t_k^\alpha$ for some t_k between s_k and $s_{n(k)}$. Thus, $|m_k - m_l| \leq C|t_k - t_l|^\alpha \leq C3^\alpha|s_k - s_l|^\alpha$ for $k \neq l$. (Here, we use the inequalities $|t_k - s_k| \leq |s_k - s_{n(k)}| \leq |s_k - s_l|$ and $|t_l - s_l| \leq |s_l - s_{n(l)}| \leq |s_k - s_l|$.) Similarly, $|m_k - (1 + \alpha)s_k^\alpha| \leq C|s_{k+1} - s_k|^\alpha$, hence Taylor's theorem provides uniform control on each term from the first sum in (33). Therefore,

$$\|\varphi\|_{B_p(\mathbb{R})}^p \leq CD^2. \quad (34)$$

Applying the extension operator \mathcal{E} from Theorem 3, the function $F = \mathcal{E}(\varphi)$ satisfies $F|_{\mathbb{R} \times \{0\}} = \varphi$ and $\|F\|_{L^{2,p}(\mathbb{R}^2)} \leq C_{SB} \|\varphi\|_{\dot{B}_p(\mathbb{R})}$. Thus, from (32),

$$S \subseteq \{(s, F(s, 0)) : s \in \mathbb{R}\}, \quad (35)$$

while from (34) we obtain

$$\|F\|_{L^{2,p}(\mathbb{R}^2)} \leq C'D^{2/p}. \quad (36)$$

We may assume that $\#S \geq 2$, for otherwise Lemma 2 is trivial. Note that $S \subseteq [0, 1]^2$ lies on a Lipschitz graph. Thus, by (35), there exists $s^* \in [0, 1]$ such that $|\partial_1 F(s^*, 0)| \leq C$. By (36) and the Sobolev theorem, $|\partial_1 F(0)| \leq C'D^{2/p}$.

Let

$$M := \max\{\|F\|_{L^{2,p}(\mathbb{R}^2)}, |\partial_1 F(0)|, 1\}.$$

Without loss of generality, by adding to F some multiple of the coordinate function $(s, t) \mapsto t$, we may assume that $\partial_2 F(0) = RM$, where $R \geq 1$ shall be determined later. This does not affect statements from the previous two paragraphs. To summarize:

$$|\partial_1 F(0)| \leq M, \quad \partial_2 F(0) = RM, \text{ and} \quad (37)$$

$$\|F\|_{L^{2,p}(\mathbb{R}^2)} \leq M, \quad \text{where } 1 \leq M \leq C'D^{2/p}. \quad (38)$$

Pick $\hat{\theta} \in C_0^\infty(\mathbb{R}^2)$ that satisfies

$$\begin{aligned} & \text{(a) } \text{supp}(\hat{\theta}) \subseteq [-1, 2]^2, \quad \text{(b) } \hat{\theta} = 1 \text{ on } [-1/2, 3/2]^2, \text{ and} \\ & \text{(c) } |\partial^\beta \hat{\theta}| \leq C, \text{ whenever } |\beta| \leq 2. \end{aligned} \quad (39)$$

Define $\hat{F} := \theta F + (1 - \theta)J_0 F$.

Mimicking the proof of (17) with help from (38),(39.a),(39.c), we obtain

$$\|\hat{F}\|_{L^{2,p}(\mathbb{R}^2)} \leq CM. \quad (40)$$

Mimicking the proof of (19) with help from (39.a),(39.b),(40), we obtain

$$|\nabla \hat{F}(y) - \nabla F(0)| \leq C'M \quad (\text{all } y \in \mathbb{R}^2).$$

Now, choose R sufficiently large, determined by p , so that the previous inequality and (37) imply that

$$|\partial_1 \hat{F}(y)| \leq CM \text{ and } \frac{RM}{2} \leq |\partial_2 \hat{F}(y)| \leq 2RM \quad (\text{all } y \in \mathbb{R}^2). \quad (41)$$

Finally, (35),(39.b) and $S \subseteq [0, 1]^2$ imply that

$$S \subseteq \{(s, \widehat{F}(s, 0)) : s \in \mathbb{R}\}. \quad (42)$$

We define $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $\Phi(s, t) = (s, \widehat{F}(s, t))$. The diffeomorphism Φ maps onto \mathbb{R}^2 because $|\partial_2 \widehat{F}|$ is bounded away from zero (see (41)).

We define $\Psi = \Phi^{-1}$. We write $\Phi = (\Phi_1, \Phi_2)$ and $\Psi = (\Psi_1, \Psi_2)$ in coordinates. As in (25), we obtain

$$\|\Psi\|_{L^{2,p}(\mathbb{R}^2)} \leq C \|\Phi\|_{L^{2,p}(\mathbb{R}^2)} \cdot \|\det(\nabla \Phi)\|_{L^\infty}^{1/p} \cdot \|(\nabla \Phi)^{-1}\|_{L^\infty}^3.$$

It follows from (40),(41) that $\|\Phi\|_{L^{2,p}(\mathbb{R}^2)} = \|\widehat{F}\|_{L^{2,p}(\mathbb{R}^2)} \leq CM$,

$\|\det(\nabla \Phi)\|_{L^\infty} \leq 2RM$ and $\|(\nabla \Phi)^{-1}\|_{L^\infty} \leq C'$. Therefore,

$$\|\Psi_2\|_{L^{2,p}(\mathbb{R}^2)} \leq \|\Psi\|_{L^{2,p}(\mathbb{R}^2)} \leq C''M^{1+1/p} \leq C''M^{3/2}. \quad (43)$$

In coordinates, $\Phi(s, t) = (s, \widehat{F}(s, t))$ and $\Psi(u, v) = (u, \Psi_2(u, v))$, where $\widehat{F}(u, \Psi_2(u, v)) = v$. Applying $\partial_2 = \frac{\partial}{\partial v}$, setting $u = v = 0$, and then using (41),

$$\partial_2 \Psi_2(0) = [\partial_2 \widehat{F}(\Psi(0))]^{-1} \geq CM^{-1}, \quad (44)$$

Finally, (42) implies that $S \subseteq \Phi(\mathbb{R} \times \{0\})$, thus we obtain

$$\Psi(S) \subseteq \mathbb{R} \times \{0\}. \quad (45)$$

Let $H = \Psi_2/\partial_2 \Psi_2(0)$. The bound $M \leq C \cdot D^{2/p}$ and (43)-(45) imply that H satisfies the conclusion of Lemma 2. This completes the proof of Lemma 2. \blacksquare

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